Gauge strata and particle generations

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Abstract

Phenomenological evidence suggests the existence of non-trivial background fields in the QCD vacuum. On the other hand SU(3) gauge theory possessses three different classes of both non-generic and non-trivial strata that may be used as classical backgrounds. It is suggested that this three-fold multiplicity of non-trivial vacua may be related to the existence of particle generations, which would then find an explanation in the framework of the standard model.

PACS: 11.15.-q, 12.38.Aw

Within the limits of its accuracy, all experimental data known so far is consistent with a $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge theory as a model for particle physics. However, several features of the observed low energy phenomena remain to be explained. Among these the generation structure, in the quark and lepton spectra, stands as one of the most intriguing puzzles.

Several schemes, going beyond the gauge group of the standard model, have been proposed to describe the existence of particle generations, ranging from family groups, horizontal symmetries, radial excitations, enlarged groups, preon models to superstrings.

Contrary to this *beyond the standard model* trend, I will argue in this letter that, by itself, the gauge group of the standard model already contains a multiplicity in the space of its solutions that is reminiscent of the particle generations structure.

For definiteness I will consider physical states as being represented by quantum fluctuations around classical solutions and physical processes as path integrals on the space of field configurations. A classical gauge theory consists of four basic objects:

(i) A principal fiber bundle $P\,(M,G)$ with structure group G and projection $\pi : P \rightarrow M$,

(ii) An affine space $\mathcal{C}$ of connections on P which, by selecting a reference connection, may be modelled by a vector space $\mathcal{A}$ of one-forms on M with values on the Lie algebra $\mathcal{G}$ of G,

(iii) The space of differentiable sections of P, called the *gauge group* Γ

(iv) A $\Gamma-$invariant functional $\mathcal{L} : \mathcal{A} \rightarrow \mathbb{R}$

All statements below refer to the case where G is a compact group. The action of Γ on $\mathcal{A}$ leads to a stratification of $\mathcal{A}$ corresponding to the classes of equivalent *orbits* $\{A^g; g \in \Gamma\}$. Let Γ_A denote the *isotropy group* of $A \in \mathcal{A}$

$$\Gamma_A = \{g \in \Gamma : A^g = A\} \tag{1}$$

The *stratum* $\Sigma\,(A)$ of A is the set of connections having isotropy groups $\Gamma-$conjugated to that of A

$$\Sigma\,(A) = \left\{B \in \mathcal{A} : \exists g \in \Gamma : \Gamma_B = g \Gamma_A g^{-1}\right\} \tag{2}$$

The configuration space of the gauge theory is the quotient space $\mathcal{A}/\Gamma$ and therefore a stratum is the set of points in $\mathcal{A}/\Gamma$ that correspond to orbits with conjugated isotropy groups.

The stratification of the gauge space when G is a compact group has been extensively studied[1] - [5]. The stratification is topologically regular. The set of strata carries a partial ordering of types, $\Sigma_\tau \subseteq \Sigma_{\tau'}$ with $\tau \leq \tau'$ if there are representatives S_τ and $S_{\tau'}$ of the isotropy groups such that $S_\tau \supseteq S_{\tau'}$. The maximal element in the ordering of types is the class of the center $Z(G)$ of G and the minimal one is the class of G itself. Furthermore $\cup_{t \geq \tau} \Sigma_t$ is open and Σ_τ is open in the relative topology in $\cup_{t \leq \tau} \Sigma_t$.

Most of the stratification results have been obtained in the framework of Sobolev connections and Hilbert Lie groups. However, for the calculation of physical quantities in the path integral formulation

$$\langle \phi \rangle = \int_{\mathcal{A}/\Gamma} \phi\,(\xi)\, e^{i\mathcal{L}(\xi)} d\mu\,(\xi) \tag{3}$$

a measure in $\mathcal{A}/\Gamma$ is required, and no such measure has been found for Sobolev connections. Therefore it is more convenient to work in a space of generalized connections $\overline{\mathcal{A}}$, defining parallel transports on piecewise smooth paths as simple homomorphisms from the paths to the group G, without a smoothness assumption[6]. The same applies to the generalized gauge group $\overline{\Gamma}$. Then, there is in $\overline{\mathcal{A}}/\overline{\Gamma}$ an induced Haar measure, the Ashtekar-Lewandowski measure[7] - [8], Sobolev connections being a dense zero measure subset of the generalized connections[9]. The question remained however of whether the stratification results derived in the context of Sobolev connections would apply to generalized connections. This question was recently settled by Fleischhack[10] who, by establishing a slice theorem for generalized connections, proved that essentially all existing stratification results carry over to the generalized connections. In some cases they even have wider generality.

Because the isotropy group of a connection is isomorphic to the centralizer of its holonomy group[11], the strata are in one-to-one correspondence with the Howe subgroups of G, that is, the subgroups that are centralizers of some subset in G. Given an holonomy group H_τ associated to a connection A of type τ, the stratum of A is classified by the conjugacy class of the isotropy group S_τ, the centralizer of H_τ

$$S_\tau = Z\left(H_\tau\right) \tag{4}$$

An important role is also played by the centralizer of the centralizer

$$H'_\tau = Z\left(Z\left(H_\tau\right)\right) \tag{5}$$

that contains H_τ itself. If H'_τ is a proper subgroup of G the connection A reduces locally to the subbundle $P_\tau = (M, H'_\tau)$. Global reduction depends on the topology of M, but it is always possible if P is a trivial bundle. H'_τ is the structure group of the *maximal subbundle* associated to type τ.Therefore the types of strata are also in correspondence with types of reductions of the connections to subbundles. If S_τ is the center of G the connection is called *irreducible*, all others are called *reducible*. The stratum of the irreducible connections is called the *generic stratum*. It is open and dense and it carries the full Ashtekar-Lewandowski measure.

Now I turn to the case $G = SU(3)$. The isotropy groups and the structure groups of the maximal subbundles are[3] :

$$\begin{array}{ccc} & \Gamma_A & H_A^{'} \\ 1 & \mathbb{Z}_3 & SU(3) \\ 2 & U(1) & U(2) \\ 3 & U(1)\times U(1) & U(1)\times U(1) \\ 4 & U(2) & U(1) \\ 5 & SU(3) & \mathbb{Z}_3 \end{array} \tag{6}$$

There are five strata. Stratum 1 is the generic stratum. All others are reducible strata. Recall now the basic assumption that physical states are represented by quantum fluctuations around classical solutions. Because of its full measure, quantum fluctuations in the path integral must be taken from the generic stratum 1. However classical solutions, around which the quantum fluctuations take place, are not required to belong to the generic stratum. For example the perturbative vacuum is in the stratum 5, that is the stratum to which the null connections ($A_\mu(x) = 0$) belong.

One-loop calculations show that the perturbative vacuum is unstable and, even more important, there is ample phenomenological evidence for the existence of non-trivial vacuum condensates in the QCD vacuum[12] [13]. This is incompatible with stratum 5 being the site for the physical vacuum. Therefore classical vacuum solutions should be looked for in the other reducible strata.

A classical solution of the gauge theory is a stationary point of the $\Gamma-$invariant functional $\mathcal{L}$, that is, a $\mathcal{L}-$critical point in $\overline{\mathcal{A}}/\overline{\Gamma}$. Gaeta and Morando[14], generalizing the classical (finite-dimensional) result of Michel[15], have proven that an orbit in $\overline{\mathcal{A}}/\overline{\Gamma}$ is critical for any $\Gamma-$invariant functional whatsoever if and only if it is isolated in its stratum. This applies of course to the most singular stratum (1), not to the other reducible strata. However using compactness arguments it is possible to prove the existence of stationary points for each particular $\Gamma-$invariant functional. Alternatively, by constructing representatives of the connections in each one of the strata 2 - 4, it is easy to check that they contain a large number of stationary points of the Yang-Mills action. Which one is the minimum energy classical solution is not important for our discussion. Consistency of the choice of a classical solution in a reducible stratum is insured by the fact that trajectories of the classical field theory remain in that same stratum[16].

In conclusion: to be compatible with non-trivial vacuum condensates in QCD, classical solutions should not be chosen in the minimal stratum (1) and we are left with a three-fold degeneracy of vacuum possibilities.

Quarks are triplet excitations over these vacua and leptons singlet excitations. Therefore a three family structure seems to emerge already at the level of the solutions of the standard model gauge group, without the need to go beyond this group.

Further insight is obtained when the gauge theory is considered as an infinite-dimensional symplectic structure, with space components of the connection and the chromoelectric field as coordinates and momenta, defined on a Cauchy surface of initial data ($x^0 = 0$ for example)[17] [18]. The Yang-Mills equations are split into evolution equations and constraints on the Cauchy surface. The solutions to the whole system form a fibre bundle over the solutions to the constraint equations. Therefore to describe the singularities of the full equations it suffices to describe those of the constraints.

The covariant derivative $\Gamma = D\overrightarrow{E}$ of the chromoelectric field on the Cauchy surface being the generator of the gauge transformations, Γ defines a momentum mapping and the set C of solutions of the constraint equations ($\Gamma = 0$) is the zero set $\Gamma^{-1}(0)$ of the momentum mapping.

In the neighborhood of a field $(\underline{\overrightarrow{A}},\underline{\overrightarrow{E}})$ without symmetries the solution set C is a manifold with tangent space given by $\ker d\Gamma$, that is

$$\overrightarrow{\nabla} \cdot \overrightarrow{e} + \left[\underline{\overrightarrow{A}}, \overrightarrow{e}\right] + \left[\overrightarrow{a}, \underline{\overrightarrow{E}}\right] = 0 \tag{7}$$

with $\overrightarrow{A} = \underline{\overrightarrow{A}} + \overrightarrow{a}$ and $\overrightarrow{E} = \underline{\overrightarrow{E}} + \overrightarrow{e}$. This is the case for fields in the generic stratum (stratum 1).

In the neighborhood of fields in all the other strata, the zero set of the momentum mapping has singularities and the solution set C is diffeomorphic to the product of a manifold and the zero set of a homogeneous quadratic function. It means that in addition to the linear constraint, Eq.(7), and a slice condition[17], fields of the same strata in the neighborhood of $(\underline{\overrightarrow{A}},\underline{\overrightarrow{E}})$ must also satisfy a quadratic constraint, which in components is

$$f_{bcd} a_c^k e_d^k = 0 \tag{8}$$

(f_{bcd} being the structure constants of G). This is a condition to be taken into account when quantizing a gauge system around fields of the non-generic strata. This additional constraint suppresses transitions to configurations of lower symmetry and, by analogy with the solutions of the Schrödinger equation on double cones[19], one expects low energy states to remain concentrated near the singularities. That is, not only the classical solutions remain in the same strata, but also low energy quantum solutions are expected to approximate and inherit the symmetries of the singular strata.

Of course, all these considerations only point out that, if the trivial vacuum is unstable, then there is a three-fold choice for non-trivial reducible backgrounds. Nothing is said about why triplet or singlet excitations over each type of vacuum have different masses. Here there are two possibilities. Part of the mass splitting may arise from the diverse nature of the vacuum classes. Remember that important algebraic and coset volume differences exist among the reducible connections. In particular the closed set nature of $\cup_{t\leq\tau}\Sigma_t$, implies that each Σ_t behaves like a boundary set for the strata of type immediately higher. Alternatively, and for the mass contributions that are not sensitive to the background, a democratic mass matrix with all elements equal might be considered, as discussed by a number of authors[20] [21] [22]. When diagonalized this leads (in leading order) to one massive state and two massless ones.

In conclusion: Whatever dynamical mechanism provides a full explanation of the mass differences between the particle generations, the fact is that the rich strata structure of the standard model must be taken into account whenever the need be felt to consider non-trivial backgrounds.

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